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Paley–Wiener Theorems on Groups of Split Rank One

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Let G be a linear semisimple Lie group of split rank one with K a maximal compact subgroup. In this paper, we consider the space $C_c^\infty(G:F)$ of all functions in $C_c^\infty(G)$ whose left and right K-translates span a finite-dimensional space. Using the analytic continuation of the principal series to define the Fourier transform, we give a characterization of the Fourier transform of the space $C_c^\infty(G:F)$. This gives an analog of the classical Paley–Wiener theorem which gives a characterization of the Fourier transform of the space $C_c^\infty(\mathbb{R}^n)$.

1. INTRODUCTION AND NOTATION

Let G be a semisimple Lie group with finite center. Let $G = KAN$ be an Iwasawa decomposition of G. That is, K is a maximal compact subgroup of G, A is a maximal vector subgroup of G with Ad A consisting of semisimple elements and A contained in the normalizer of N, a maximal simply connected nilpotent subgroup of G. Let M be the centralizer of A in K and M' the normalizer of A in K. We denote the Lie algebras of G, K, A, N, and M by $\mathfrak{g}$, $\mathfrak{k}$, $\mathfrak{a}$, $\mathfrak{n}$, and $\mathfrak{m}$, respectively. Now exp: $\mathfrak{a} \to A$ is a Lie isomorphism and we denote its inverse by log. Now, if $g \in G$, g can be written uniquely as

$$g = k(g \exp H(g)\, n(g)$$

where $k(g) \in K$, $H(g) \in \mathfrak{a}$ and $n(g) \in N$. Furthermore, there is an involution $\theta: G \to G$ such that $K = \{g: \theta(g) = g\}$. Set $N = \theta(N)$ and denote its Lie algebra by $\bar{\mathfrak{n}}$.

For $\alpha \in \mathfrak{a}^*$ let $\mathfrak{g}_\alpha = \{X \in \mathfrak{g}: [HX] = \alpha(H)X$ for all $H \in \mathfrak{a}\}$. Let

$$\Delta = \{\alpha \in \mathfrak{a}^* \sim \{0\}: \mathfrak{g}_\alpha \neq (0)\}.$$

We then have

$$\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{a} \oplus \sum_{\alpha \in \Delta} \mathfrak{g}_\alpha = \mathfrak{m} \oplus \mathfrak{a} \oplus \mathfrak{n} \oplus \bar{\mathfrak{n}}.$$

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Moreover, there is a subset P of Δ such that

$$\mathfrak{n} = \sum_{\alpha \in P} \mathfrak{g}_\alpha \qquad \text{and} \qquad \bar{\mathfrak{n}} = \sum_{\alpha \in P} \mathfrak{g}_\alpha \,.$$

For $\alpha \in \mathfrak{a}^*$ let $m_\alpha = \dim G_\alpha$ and $2\rho = \sum_{\alpha \in P} m_\alpha \alpha$.

Besides the Iwasawa decomposition of G we will have occasion to use the following decomposition of G. Let $\mathfrak{a}^+ = \{H \in \mathfrak{a} \colon \alpha(H) > 0 \text{ for all } \alpha \in P\}$ and let $A^+ = \exp \mathfrak{a}^+$. Then $G = K\overline{A^+}K$ where A^+ is the closure of $\overline{A^+}$. This is called the polar decomposition of G.

Let τ be a double representation of K on a finite-dimensional Hilbert space V (i.e., K acts on the left and right of V by means of τ). Let $C_c^\infty(G, \tau)$ be the set of functions $f \colon G \to V$ which are C^∞ with compact support and have the property that $f(k_1 g k_2) = \tau(k_1)\, f(g)\, \tau(k_2)$ $(k_1, k_2 \in K)$.

Now for $\omega \in \hat{M}$, the unitary dual of M, $\nu \in A_{\mathbf{C}}^*$ and for $f \in C_c^\infty(G, \tau)$ we define

$$\psi_f(\omega, \nu) = \int_M g_f(m)\, \overline{\chi_\omega(m)}\, dm$$

where χ_ω is the character of ω and where

$$g_f(m) = \int_A \int_N f(man)\, e^{-(i\nu - \rho)(\log a)}\, dn\, da.$$

Observe that $\psi_f(\omega, \nu) \in V^M = \{v \in V \colon \tau(m)v = v\tau(m)\}$ for all $m \in M$.

Now for $\tilde{A} \in V^M$ we define the Eisenstein integral of Harish-Chandra as

$$E(A : \nu : x) = \int_K \tau(k(xk))\, \tilde{A}\tau(k)^{-1}\, e^{(i(i\nu - \rho)H(ak)}\, dk.$$

Our main results will deal with groups G for which $\dim \mathfrak{a} = 1$ (G has split rank one). In any case we have from Harish-Chandra [4] that there exists a nonnegative function $\mu(\omega, \nu)$ on $\hat{M} \times \mathfrak{a}^*$ with the following properties.

(1) For a fixed $\omega \in \hat{M}$ the function $\nu \to \mu(\omega, \nu)$ extends to a meromorphic function on $\mathfrak{a}_{\mathbf{C}}^*$ and $\mu(\omega, \nu) = \mu(s\omega, s\nu)$ for all $s \in W$ where $W = M'/M$ the restricted Weyl group of G. (Here W acts on $\mathfrak{a}$ and hence on $\mathfrak{a}_{\mathbf{C}}^*$, and for $\omega \in \hat{M}$ $s\omega$ is the element of $\hat{M}$ defined by $s\bar{\omega}(m) = \bar{\omega}(w^{-1}mw)$ $(m \in M, w \in W, \bar{\omega} \in \omega)$).

(2) If $f \in C_c^\infty(G, \tau)$ the function

$$f(g) - \sum_{\omega \in \hat{M}} \int_{A^*} E(\psi_f(\omega, \nu) : \nu : g)\, \mu(\omega, \nu)\, d\nu$$

is a cusp form on G in the sense of Harish-Chandra [4].

In Section 2 we review some results of Harish-Chandra, and extend some results of R. Gangolli and S. Helgason. In Section 3 we obtain some results on generalized C-functions which will be of use to us in Section 4.

Finally, in Section 4 we prove an analog of the classical Paley–Wiener theorem for linear Lie groups of split rank one.

The results of this paper grew out of conversations with R. Gangolli and S. Helgason, and I would like to express my appreciation to them. I would also like to express my thanks to Nolan Wallach for many valuable suggestions and his patience in reading my original version.

2. Differential Equations on Lie Groups

Let V and τ be as in Section 1 and let $\tilde{A} \in V^M$. Fix $a \in A^+$ and consider the function $E(\tilde{A} : \nu : a)$. In this section we obtain an explicit expansion for the function $E(\tilde{A} : \nu : a)$ which is due to Harish-Chandra. We also extend infinitesimally some results of Gangolli and Helgason. Except for minor changes these results may be found in Warner [13].

It is not hard to see that $E(\tilde{A} : \nu : x)$ is $\mathscr{Z}$-finite where $\mathscr{Z}$ is the center of $U(\mathfrak{g})$, the complexification of the enveloping algebra $\mathfrak{g}$. Of particular importance to us is the action of Γ, the Casimir operator of $\mathfrak{g}$, on $E(\tilde{A} : \nu : x)$.

Let B denote the Killing form of $\mathfrak{g}$ and for each $\alpha \in P$ let $X_{\alpha,1},..., X_{\alpha,m_\alpha}$ be a basis of $\mathfrak{g}_\alpha$ for which $B(X_{\alpha,i}, \theta(X_{\alpha,j})) = -\delta_{ij}$. Select a basis $H_1,..., H_\ell$ of $\mathfrak{a}$ for which $B(H_i, H_j) = \delta_{ij}$, and let $U_1,..., U_r$ be a basis of $\mathfrak{m}$ such that $B(U_i, U_j) = -\delta_{ij}$. Then

$$\Gamma = \sum_{i=1}^{l} H_i^2 - \sum_{i=1}^{r} U_i^2 - \sum_{\alpha \in P} \sum_{i=1}^{m_\alpha} [X_{\alpha,i}\theta(X_{\alpha,i}) + \theta(X_{\alpha,i}) X_{\alpha,i}].$$

Set $\nabla = -\sum_{i=1}^{r} U_i^2$, $Y_{\alpha,i} = \frac{1}{2}(X_{\alpha,i} + \theta(X_{\alpha,i}))$, and for $\nu \in \mathfrak{a}^*$ let H_ν be the element of $\mathfrak{a}$ for which $B(H, H_\nu) = \nu(H)$ for all $H \in \mathfrak{a}$. Suppose now that $a = \exp H$ where $H \in \mathfrak{a}$ and $\alpha(H) \neq 0$ for any $\alpha \in P$. Then

$$(*) \quad \Gamma = \sum_{i=1}^{2} H_i^2 + \sum_{\alpha \in P} m_\alpha \coth \alpha(H) H_\alpha + \nabla$$
$$- \sum_{\alpha \in P} \frac{2}{\sinh_\alpha^2(H)} \sum_{i=1}^{m_\alpha} [Y_{\alpha,i}^2 - 2 \cosh \alpha(H) \operatorname{Ad} a^{-1}(Y_{\alpha,i}) Y_{\alpha,i}$$
$$+ \operatorname{Ad} a^{-1}(Y_{\alpha,i})^2]$$

Now if $f \in C^\infty(G, \tau)$ and $a = \exp H$ as above we obtain by (*) that

$$\begin{aligned}(\Gamma f)(a) &= (\Gamma f)\,|_A\,(a) = \tilde{\Gamma})f\,|_A)(a) \\ &= \sum_{i=1}^{l} H_i^2 f(a) + \sum_{\alpha\in P} m_\alpha \coth a(H)\, H_\alpha f(a) + \nabla f(a) \\ &\quad - \sum_{\alpha\in P} \frac{2}{\sinh^2 a(H)} \sum_{i=1}^{m_\alpha} [f(a)\,\tau(Y_{\alpha,i})^2 - 2\cosh \alpha(H)\,\tau(Y_{\alpha,i}) \\ &\quad + f(a)\,\tau(Y_{\alpha,i})^2]\end{aligned}$$

where for $X \in \mathfrak{g}$ $Xf(x) = (d/dt) f(x \exp tX)\,|_{t=0}$. The operator $\tilde{\Gamma}$ is called the radial component of Γ. More generally, for $z \in \mathscr{Z}$ there is a differential operator $\tilde{z}$, the radial component of z, defined on $A' = A \sim \{1\}$ such that $(zf)\,|_{A'} = \tilde{z}(f\,|_{A'})$ for $f \in C^\infty(G, \tau)$. In the following we will call a function F from A' or A^+ to $\mathrm{End}(V^M)$ $\mathscr{Z}$-finite if the linear span of all zF for $z \in \mathscr{Z}$ is finite dimensional.

For $\mu, \nu \in \mathfrak{a}^*$ let $\langle \mu, \nu\rangle = B(H_\mu, H_\nu)$ and extend $\langle\ ,\ \rangle$ to $\mathfrak{a}_{\mathbb{C}} \times \mathfrak{a}_{\mathbb{C}}$ by making $\langle\ ,\ \rangle$ complex linear in each variable. Then we obtain

$$\Gamma E(A : \nu : x) = (-\langle \nu, \nu\rangle - \langle \rho, \rho\rangle)\, E(\tilde{A} : \nu : x) + E(\tau(\nabla)\tilde{A} : \nu : x).$$

Let L be the set of all points in $\mathfrak{a}^*$ of the form $\sum_{\alpha\in P} n_\alpha \alpha$ where each n_α is a non negative integer. Letting $a = \exp H$ as above we wish to find for each $s \in W$ and each $\mu \in L$ a function Γ_μ on $\mathfrak{a}_{\mathbb{C}}^*$ with values in $\mathrm{End}(V^M)$ such that:

(a) $E_s(a : \nu) = \sum_{\mu\in L} \Gamma_\mu(is\nu - \rho)\, e^{(is\nu-\rho-\mu)(H)}$ defines a $\mathscr{Z}$-finite function from A^+ to $\mathrm{End}(V^M)$ which is meromorphic in ν; and,

(b) $E(\tilde{A} : \nu : a)$ may be expressed as a certain finite linear combination of the $E_s(a : \nu)$.

From (*) we see that the $\Gamma_\mu(i\nu - \rho)$ satisfy the following recursion formula:

$$\begin{aligned}(**)\quad &(\langle \mu, \mu\rangle - 2i\langle \mu, \nu\rangle)\, \Gamma_\mu(i\nu - \rho) + [\tau(\nabla), \Gamma_\mu(i\nu - \rho)] \\ &= - \sum_{\alpha\in P} \sum_{n=1} 2m_\alpha \langle i\nu - \rho - (\mu - 2n\alpha), \alpha\rangle\, \Gamma_{\mu-2n\alpha}(i\nu - \rho) \\ &\quad + \sum_{\alpha\in P} 8 \sum_{n\geqslant 1} (n+1) \sum_{i=1}^{m_\alpha} [\tau(Y_{\alpha,i})^2 \Gamma_{\mu-2(n+1)\alpha}(i\nu - \rho) \\ &\quad + \Gamma_{\mu-2(n+1)\alpha}(i\nu - \rho)\, \tau(Y_{\alpha,i})^2 - \tau(Y_{\alpha,i})(\Gamma_{\mu-(2n+1)\alpha}(i\nu - \rho) \\ &\quad + \Gamma_{\mu-(2n+3)\alpha}(i\nu - \rho))\, \tau(Y_{\alpha,i})].\end{aligned}$$

Here we caution the reader that for $Y \in K$ and $u \in V^M$ we set

$$(\Gamma_\mu(iv - \rho)\, \tau(Y))(u) = (\Gamma_\mu(iv - \rho)(u))\, \tau(Y).$$

Here we assume $\Gamma_0(i\nu - \rho) = I$ and $\Gamma_\mu = 0$ if $\mu \notin L$.

Let $\lambda_1, \ldots, \lambda_r$ be the eigenvalues of the operator $\tau(\nabla)$ on V^M and let $\gamma_{ij} = \lambda_i - \lambda_j$. Clearly, we may solve our recursion formulas provided no $\langle \mu, \mu \rangle - 2i\langle \mu, \nu \rangle + \gamma_{ij} = 0$ for any $\mu \in L$, and in this case Harish-Chandra has shown that the functions $E_s(a : \nu)$ are $\mathscr{Z}$-finite. (See Warner [13].)

Let $H_0 \in \mathfrak{a}$ and write $-i\nu = \zeta + i\eta$ where $\zeta, \eta \in \mathfrak{a}^*$. Set $T(H_0) = \{\nu \in \mathfrak{a}_{\mathbb{C}}^* : H_\zeta \in H_0 + \mathfrak{a}^+\}$ and for $1 \leqslant k, j \leqslant r$ consider the function $f_{j,k} : L \times \mathfrak{a}_{\mathbb{C}}^* \to \mathbb{C}$ defined by

$$f_{j,k}(\mu, \nu) = \langle \mu, \mu \rangle - 2i\langle \mu, \nu \rangle + \gamma_{j,k}.$$

We now obtain easily the following lemma.

LEMMA 2.1. *There are at most finitely many μ's in L for which $f_{j,k}(\mu, \nu) = 0$ for $\nu \in T(H_0)$. Let L_0 be the set of these μ's and let*

$$P(\nu) = \prod_{j,k=1}^{r} \prod_{\mu \in L_0} f_{j,k}(\mu, \nu).$$

Then for all $\mu \in L$, $\nu \to P(\nu) L_\mu(i\nu - \rho)$ is regular for $\nu \in T(H_0)$.

When all the $\Gamma_\mu(is\nu - \rho)$'s are defined Harish-Chandra has shown that there exist $C(s : \nu) \in \operatorname{End}(V^M)$ for which

$$E(\tilde{A} : \nu : a) = \sum_{s \in W} E_s(a : \nu)\, C(s : \nu)(\tilde{A}).$$

For our purposes, it is convenient to examine the function $\Phi(a; \nu) = \delta^{1/2}(a)\, E_1(a : \nu)$ where 1 is the identity of W and where

$$\delta(a) = \prod_{\alpha \in P} (\sinh \alpha(H))^{m_\alpha}.$$

(This was first done by Gangolli [2].) We now see that

$$\Phi(a; \nu) = \textstyle\sum_{\mu \in L} a_\mu(\nu)\, e^{(i\nu - \mu)(H)}$$

where $a_\mu(\nu)$ satisfies the equation

$$\begin{aligned}
(***) \quad & [(\langle \mu, \mu \rangle - 2i\langle \mu, \nu \rangle]\, a_\mu(\nu) + [\tau(\nabla), a_\mu(\nu)] \\
&= \sum_{\alpha \in P} \sum_{n \geqslant 0} 8(n+1) \sum_{i=1}^{m_\alpha} [\tau(Y_{\alpha,i})^2 a_{\mu - 2(n+1)\alpha}(\nu) + a_{\mu-2(n+1)\alpha}(\nu)\, \tau(Y_{\alpha,i})^2 \\
&\quad - \tau(Y_{\alpha,i})(a_{\mu-(2n+1)\alpha}(\nu) + a_{\mu-(2n+3)\alpha}(\nu))\tau(Y_{\alpha,i})] \\
&\quad - \sum_{\alpha,\beta \in P} \frac{m_\alpha m_\beta}{4} \langle \alpha, \beta \rangle \sum_{\substack{j,k \\ j+k \geqslant 1}} d_j d_k a_{\mu - 2j\alpha - 2k\beta}(\nu) \\
&\quad + \sum_{\alpha \in P} 2m_\alpha \langle \alpha, \alpha \rangle \sum_{n \geqslant 0} (n+1)\, a_{\mu - 2(n+1)\alpha}(\nu).
\end{aligned}$$

Here $d_0 = 1$, $d_j = 2$ for $j \geqslant 1$, $a_\lambda(\nu) = 0$ if $\lambda \in L$, and $a_0(\nu) = I$.

Let $P(\nu)$ be as in Lemma 2.1 and consider for each $\mu \in L$ $\| P(\nu)\, a_\mu(\nu)\|$ where $\| \ \|$ denotes the operator norm of $P(\nu)\, a_\mu(\nu)$.

LEMMA 2.2 (see [2]). *Fix $H_0 \in \mathfrak{a}$ and $H_1 \in \mathfrak{a}^+$. Then there is a $Q(\nu) > 0$ of polynomial growth depending on H_0 and H_1 such that*

$$\| P(\nu)\, a_\mu(\nu)\| \leqslant Q(\nu)\, e^{\mu(H_1)}$$

for all $\nu \in T(H_0)$ and all $\mu \in L$.

Proof ([see 6]). Let $I_\mu(\nu) \in \operatorname{End}(\operatorname{End} V^M)$ be the operator defined by

$$I_\mu(\nu)(T) = (\langle \mu, \mu\rangle - 2i\langle \mu, \nu\rangle)T + [\tau(\nabla), T].$$

Observe that there is a $C > 0$ and a $B > 0$ such that if $\langle \mu, \mu\rangle > C$ and $\nu \in T(H_0)$, $\| I_\mu(\nu)^{-1} \| \leqslant B/\langle \mu, \mu\rangle$ where $\| \ \|$ again denotes operator norm. Set $m = \max_{\alpha\in P} m_\alpha$ and select $A > 0$ so that $\| \tau(Y_{\alpha,i})^2 \| \leqslant A$ and $\| \tau(Y_{\alpha,i})\| \leqslant A$ for all α and i. Then if $\langle \mu, \mu\rangle > C$ we have

$$\begin{aligned}\| P(\nu)\, a_\mu(\nu)\| \leqslant \frac{B}{\langle \mu, \mu\rangle} \Big\{ &\sum_{\alpha\in P} \sum_{n\geqslant 0} m(n+1)A \Big[2 \| P(\nu)\, a_{\mu-2(n+1)\alpha}(\nu)\| \\ &+ \| P(\nu) a_{\mu-(2n+3)\alpha}(\nu) \| \Big] + \sum_{\alpha,\beta} m^2\langle \alpha, \beta\rangle \sum_{\substack{j,k\\ j+k\geqslant 1}} \| P(\nu)\, a_{\mu-2j\alpha-2k\beta}(\nu) \| \\ &+ \sum_{\alpha} 2m\langle \alpha, \alpha\rangle \sum_{n\geqslant 0} (n+1) \| P(\nu)\, a_{\mu-2(n+1)\alpha}(\nu)\| \Big\}.\end{aligned}$$

There is a $C_0 > 0$ such that the first and third double series are dominated by $\sum_{\lambda\in L} C_0 \| \lambda \| \| P(\nu)\, a_{\mu-\lambda}(\nu)\|$ and the second double series is dominated by $\sum_{\lambda\in L} C_0 \| P(\nu)\, a_{\mu-\lambda}(\nu)\|$. So if $\langle \mu, \mu\rangle > C$

$$\| P(\nu)\, a_\mu(\nu)\| \leqslant \frac{BC_0}{\langle \mu, \mu\rangle} \sum_{s\in L} (2 \| \lambda \| + 1) \| P(\nu)\, a_{\mu-\lambda}(\nu)\|.$$

Now, since there are only finitely many $\mu \in L$ for which $\langle \mu, \mu\rangle \leqslant C$, by the definition of $a_\mu(\nu)$ we have a function $Q(\nu) > 0$ of polynomial growth for which $\| P(\nu)\, a_\mu(\nu)\| \leqslant Q(\nu)$ if $\langle \mu, \mu\rangle \leqslant C$. Assume now that $\| P(\nu)\, a_\mu(\nu)\| \leqslant Q(\nu)$ for all μ with $\langle \mu, \mu\rangle \leqslant BC_0N_0$ where $N_0 = \sum_{\lambda\in L} (2 \| \lambda \| + 1)\, e^{-\lambda(H_1)} < \infty$. Suppose now that $\langle \mu, \mu\rangle > \max(C, BC_0N_0)$ and

$$\| P(\nu)\, a_{\mu-\lambda}(\nu)\| \leqslant Q(\nu)\, e^{(\mu-\lambda)(H_1)}$$

for all $\lambda \in L \sim \{0\}$. Then

$$\| P(\nu)\, a_\mu(\nu) \| \leqslant \frac{BC_0 Q(\nu)}{\langle \mu, \mu \rangle} \sum_{s \in L} (2 \| \lambda \| + 1)\, e^{(\mu - \lambda)(H_1)}$$

$$= \frac{BC_0 N_0}{\langle \mu, \mu \rangle} Q(\nu)\, e^{\mu(H_1)} \leqslant Q(\nu)\, e^{\mu(H_1)}$$

which proves our result.

LEMMA 2.3 (see [6]). *For $\nu \in T(H_0)$ there is a function $Q_0(\nu) > 0$ of polynomial growth depending on H_0 and H_1 such that*

$$\| P(\nu)\, \Gamma_\mu(i\nu - \rho) \| \leqslant Q_0(\nu)\, e^{\mu(H_1)}.$$

Proof. Now $\delta(a)^{-1/2} = e^{-\rho(H)} \sum_{\mu \in L} b_\mu e^{-\mu(H)}$ where b_μ is of polynomial growth in μ. Thus

$$\Gamma_\mu(i\nu - \rho) = \sum_s b_\lambda a_{\mu - \lambda}(\nu).$$

Hence $\| P(\nu) L_\mu(i\nu - \rho) \| \leqslant Q(\nu)\, e^{\mu(H_1)} \sum_\lambda | b_\lambda |\, e^{-\lambda(H_1)}$. Taking

$$N \geqslant \textstyle\sum_{\lambda \in L} | b_\lambda |\, e^{-\lambda(H_1)}$$

and $Q_0(\nu) = NQ(\nu)$ we have our result.

The proofs of Lemmas 2.2 and 2.3 with minor changes are taken from the proof of Theorem 2.4 of Helgason [6].

Now for $\nu \in \mathfrak{a}_{\mathbb{C}}^*$, $\nu = \zeta + i\xi$ where $\zeta, \xi \in \mathfrak{a}^*$. Let $\{\alpha_1, \alpha_2, \ldots, \alpha_\ell\}$ be the set of simple roots in P. That is, if $\alpha \in P$ $\alpha = \sum_{i=1}^{\ell} n_i \alpha_i$ where each n_i is a nonnegative integer.

Keeping our notation as above and fixing $-H_0 \in \mathfrak{a}^*$ we now prove

LEMMA 2.4. *There is an $\eta > 0$ and a function $Q_0(\nu) > 0$ of polynomial growth such that for $\| \xi \| \leqslant \eta$*

$$\left\| e^{\rho(H)} E(\hat{A} : \nu : a) - \sum_{s \in W} C(s : \nu)(\hat{A})\, e^{is\nu(H)} \right\|$$

$$\leqslant Q_0(\nu) \sum_{s \in W} \left\| \frac{C(s : \nu)(\hat{A})}{P(s\nu)} \right\| e^{-s\xi(H)} \prod_{i-1}^{l} \frac{e^{-\alpha_i(H - H_1)}}{1 - e^{-\alpha_i(H - H_1)}}$$

for all $a = \exp H$ with $H - H_1 \in \mathfrak{a}^+$.

Proof. Since $-H_0 \in \mathfrak{a}^+$ we may find an $\eta > 0$ for which $s\nu \in T(H_0)$ for all $s \in W$ as long as $\| \xi \| \leqslant \eta$. Then

$$\begin{aligned}
&\Big\| e^{\rho(H)} E(\hat{A} : \nu : a) - \sum_{s\in W} C(s:\nu)(\hat{A})\, e^{is\nu(H)} \Big\| \\
&= \Big\| \sum_{s\in W} \sum_{\mu\in L} \Gamma_\mu(is\nu - \rho)\, C(s:\nu)(\hat{A})\, e^{(is\nu-\mu)(H)} \Big\| \\
&\leqslant \sum_{s\in W} \Big\| \frac{C(s:\nu)(\hat{A})}{P(s\nu)} \Big\| \, \| P(s\nu)\, \Gamma_\mu(is\nu - \rho) \| \, e^{-(s\xi+\mu)(H)} \\
&\leqslant \sum_{s\in W} \Big\| \frac{C(s:\nu)(\hat{A})}{P(s\nu)} \Big\| \, e^{-s\xi(H)} Q(s\nu) \sum_{\mu\in L} e^{-\mu(H-H_1)} \\
&\leqslant \sum_{s\in W} \Big\| \frac{C(s:\nu)(\hat{A})}{P(s\nu)} \Big\| \, e^{-s\xi(H)} Q(s\nu) \prod_{i=1}^{l} \frac{e^{-\alpha_i(H-H_1)}}{1 - e^{\alpha_i(H-H_1)}} .
\end{aligned}$$

Selecting $Q_0(\nu) \geqslant \max_{r\in W} Q(s\nu)$ of polynomial growth we have our result.

3. Generalized C-Functions

In this section, using some results of Schiffmann [11], we exhibit some properties of C-functions which will be of use to us in the future. We now assume that G is a linear Lie group of split rank one.

Let τ and V be as in Section 2. We now define for $\nu \in \mathfrak{a}_{\mathbb{C}}^*$ two operators $j^+(\nu)$, $j^-(\nu)$ in $\operatorname{End}(V^M)$ as follows:

$$j^*(\nu) = \int_N e^{-(i\nu+\rho)(H(n))} \tau((k(\bar{n}))^{-1}\, d\bar{n};$$

$$j^-(\nu) = \int_N e^{(i\nu-\rho)(H(n)} \tau((k(\bar{n}))\, d\bar{n}.$$

Here we must remark that the above integrals converge only for ν in a tube and that for no ν do both integrals converge. However, Schiffmann [11] has shown that these integrals define operators which extend meromorphically to all of $\mathfrak{a}_{\mathbb{C}}^*$.

We now define the operators $C(s : \nu)$ mentioned in Section 2. For $\mu \in V^M$ set $C(1 : \nu)(u) = (u) j^+(\nu)$ and $C(-1 : \nu)(u) = \tau(w) j^-(\nu)(u)\, \tau(w)^{-1}$ where $w \in -1 \in W$. Set also $\bar{C}(1 : \nu)(u) = (u) j^-(\nu)$ and $\bar{C}(-1 : \nu)(u) = j^+(\nu)(\tau(w)^{-1}(u)\, \tau(w))$. Observe $C(1 : \nu)^* = \bar{C}(1 : \nu)$ and $C(-1 : \nu)^* = \bar{C}(-1 : \nu)$ for $\nu \in \mathfrak{a}^*$.

We now recall that G is a linear Lie group of split rank one. In this case, Schiffmann [11] has shown that the matrix entries of $C(1 : \nu)$ are finite linear combinations of functions of the form

$$\frac{\Gamma((i\nu + \mu + \rho - 2s)/4)}{\Gamma((i\nu + \mu + \rho)/4)} \cdot \frac{\Gamma((i\nu + \mu - 2s - 2r)/2)}{\Gamma((i\nu + \mu + m_\alpha)/2)}$$

if 2α is a root where $\nu = \nu\alpha$, $\rho = m_\alpha + 2m_{2\alpha}$; μ, r, and s are integers with $2r + 2s \leqslant \mu$. If 2α is not a root the matrix entries of $C(1 : \nu)$ are finite linear combinations of functions of the form

$$\frac{\Gamma((i\nu + \mu - 2r)/2)}{\Gamma((i\nu + \mu + m_\alpha)/2)}.$$

Remark. As pointed out by Schiffmann [11], we have that $C(1 : \nu)$ is a rational function of ν if 2α is not a root and m_α is even.

LEMMA 3.1. *Let $0 < a < 1$, $0 < b \leqslant 1$, and $\delta > 0$. Then for n a positive integer, $|\arg z| < \pi - \delta$, and $|z|$ large there exists a polynomial $Q(z)$ of degree n such that*

$$\Gamma(z + a)/\Gamma(z + b) = z^{a-b}(Q(1/z) + O(z^{-n-1/2})).$$

Proof. From Whittaker and Watson [14, p. 278] we have for $|\arg z| < \pi - \delta$ and $|z|$ large

$$\log \Gamma(z + a) = (z + a - \tfrac{1}{2}) \log z - z + \tfrac{1}{2} \log 2\pi + \sum_{m=1}^{n} \frac{C_m(a)}{z^m} + O(z^{-n-1/2}).$$

Thus,

$$\log \Gamma(z + a) - \log \Gamma(z + b) = (a - b) \log z + \sum_{m=1}^{n} \frac{C_m(a) - C_m(b)}{z^m} + O(z^{-n-1/2}).$$

Let

$$P(z) = \sum_{m=1}^{n} \frac{C_m(a) - C_m(b)}{z^m}$$

Then we have

$$\log \frac{\Gamma(z + a)\, z^{b-a}}{\Gamma(z + b)\, e^{P(z)}} = O(z^{-n-1/2}).$$

As $|e^w - 1| \leqslant e^{|w|} - 1$ we have that

$$\frac{\Gamma(z + a)}{\Gamma(z + b)} \frac{z^{b-a}}{e^{P(z)}} = 1 + O(z^{-n-1/2}).$$

So

$$\Gamma(z + a)/\Gamma(z + b) = z^{b-a} e^{P(z)}(1 + O(z^{-n-1/2})).$$

Now there is a polynomial $Q(z)$ of degree n such that $e^{P(z)} = Q(1/z) + O(z^{-n-1})$ and our result follows.

We now consider $\det C(1 : \nu) = D(\nu)$ and set $d(z) = D(-iz)$.

LEMMA 3.2. *For $\delta > 0$ there is an integer $N > 0$ and a constant $R > 0$ such that for $|\arg z| \leqslant \pi - \delta$ and $|z| \geqslant R$*

$$|d(z)|^{-1} \leqslant |z|^N.$$

Proof. From Harish-Chandra [4] we know that $d(z)^{-1} \not\equiv 0$. Now $d(z)^{-1}$ is a finite sum of products of sums of the form

$$r(z)\,\Gamma\left(\frac{z+a}{4}\right)\Big/\Gamma\left(\frac{z+b}{4}\right) \quad \text{or} \quad \Gamma\left(\frac{z+c}{2}\right)\Big/\Gamma\left(\frac{z+f}{2}\right)$$

where $r(z)$ is rational and $0 < a/4, b/4, c/2, d/2 \leqslant 1$. By Lemma 3.1 we see that there is a $k > 0$ such that

$$d(z)^{-1} = \frac{P(z)}{\sum_{i=1}^{m} C_i z^{\alpha_i} + O(z^{-k})}$$

where $P(z)$ is a polynomial and $\alpha_1 > \alpha_2 > \cdots > \alpha_m > k$ for $|\arg z| \leqslant \pi - \delta$ and $|z|$ large and our result now follows.

COROLLARY 3.3. *Fix α a real number. Then for* $\operatorname{Re} z \geqslant \alpha$, *$d(z)$ has only finitely many zeros.*

Proof. This is immediate from Lemma 3.2 and the fact that $d(z)^{-1}$ is meromorphic.

Remark. The asymptotic formula used in Lemma 3.1 is in some sense a replacement for the condition "$d(z)^{-1}$ is meromorphic at infinity."

COROLLARY 3.4. *For $\delta > 0$ there is an $R > 0$ and an $N > 0$ such that for $|\arg z| \leqslant \pi - \delta$ and $|z| > R$ the matrix entries of $C(1 : -iz\alpha)^{-1}$ are bounded in absolute value by $|z|^N$.*

Proof. As $C(1 : -iz\alpha)^{-1} = d(z)^{-1} \operatorname{Adj} C(1: -iz\alpha)$ our result follows from Lemmas 3.1 and 3.2.

Using the adjoint formulas relating j^+ and j^- we obtain similar estimates on the other C-functions.

We conclude this section with the following lemma which will be useful to us in the next section. Although it has been proved by Harish-Chandra in much greater generality we include its proof at the request of the referee.

LEMMA 3.5. *Fix $\omega \in \hat{M}$ and suppose that $\nu \to F(\omega, \nu)$ has the property that $(1 + \|\nu\|)^n \|F(\omega, \nu)\|$ is bounded on $\mathfrak{a}^*$ for all integers n. Then the function*

$$f(x) = \int_{\mathfrak{a}^*} E(F(\omega, \nu) : \nu : x)\, \mu(w, \nu)\, d\nu$$

is C^∞ in G.

Proof. From Lemma 3.1, the Maass–Selberg relations of Harish-Chandra and the fact that $\mu(\omega, \nu)$ is holomorphic on $\mathfrak{a}^*$ (see [4]) we have that

$$\mu(\omega, \nu) \leqslant C(1 + \|\nu\|)^m \qquad (\nu \in \mathfrak{a}^*)$$

for some integer $m > 0$.

Setting $A(\nu; x, k) = \tau(k(xk) F(w, \nu)\, \tau(k)^{-1} e^{(i\nu-\rho)(H(xk))}$ and using the compactness of K we see that for $D \in U(\mathfrak{g})$

$$DE(F(\omega, \nu) : \nu : x) = \int_K D_x A(\nu, x, k)\, dk$$

where D_x denotes differentiation in the x-variable. Now $D_x A(\nu, x, k)$ is a finite linear combination of terms of the form

$$a(\nu, x, k) = L(x, k)\, F(w, \nu)\, \tau(h)^{-1} \prod_{s=1}^{n} (i\nu - \rho)(H_r(xk))\, e^{(i\nu-\rho)(H(xk))}$$

where $L\colon G \to \operatorname{End} V$ and $H_r\colon G \to \mathfrak{a}(r \leqslant n)$ are C^∞. From the hypothesis on F and the above estimate on $\mu(\omega, \nu)$ we see that $a(\nu, x, k)$ is integrable on $\mathfrak{a}^* \times K$ and C^∞ on $G \times K$. Now set

$$a(x) = \int_{\mathfrak{a}^*} \int_K a(\nu, x, k)\, dk\mu(\omega, \nu)\, d\nu.$$

In order to complete our proof it suffices to show that for $X \in \mathfrak{g}$

$$Xa(x) = \int_{\mathfrak{a}^*} \int_K X_x a(\nu, x, h)\, dk\mu(\omega, \nu)\, d\nu.$$

Set $\phi_t(x) = \exp\text{-}tXx$ and consider

$$\left| \frac{1}{t} (a(\nu, \phi_t(x), k) - a(\nu, x, k) - tX_x a(\nu, x, k)) \right| = a(\nu, t, x, k).$$

By our assumption on F our result will be proved if we show that this function for a fixed X is dominated by continuous function of the form $\|F(w, \nu)\|\, P(t, \nu)$ where $P(t, \nu)$ where $P(t, \nu)$ has polynomial growth in ν and $P(0, \nu) = 0$.

Since L is C^∞ we have that

$$L(\phi_t(x)k) - L(xk) - tXL(xk) = O(t^2)$$

uniformly in k and setting

$$H(\nu, x, k) = \prod_{r=1}^{m} (i\nu - \rho)(H_r(xk))$$

we have that

$$| H(\nu, \phi_t(x), k) - H(\nu, x, k) - tX_x H(\nu, x, k)| \leqslant t^2(1 + \|\nu\|)^l$$

for some integer l.

We now consider the function

$$| e^{(i\nu-\rho)H(\phi_t(x)k)} - e^{(i\nu-\rho)H(xk)} - t(i\nu - \rho)(XH(xk))\, e^{(i\nu-\rho)(H(xk))} |$$

which for X fixed is

$$\leqslant A \mid e^{(i\nu-\rho)(H(\phi_t(x)k)-H(xk))} - 1 - t(i\nu - \rho)(XH(xk))|$$
$$= A \mid e^{(i\nu-\rho)(tXH(xk)+0(t^2))} - 1 - t(i\nu - \rho)(XH(xk))|$$
$$\leqslant A_1 \mid e^{(i\nu-\rho)(0(t^2))} - 1 \mid + A \mid e^{(i\nu-\rho)t(XH(xk))} - 1 - t(i\nu - \rho)(XH(xk))|$$

for some A_1, $A > 0$. That this term is $\leqslant Bt^2(1 + \| \nu \|^2)$ for some $B > 0$ now follows from the following facts:

$$| \cos x - 1 | \leqslant Cx^2 \qquad \text{and} \qquad | \sin x - x | \leqslant Cx^2$$
$$\text{for some } C > 0 \text{ and all real } x.$$

By combining our above inequalities it follows that for a fixed X there is a continuous function $P(t, \nu)$ of polynomial growth in ν for which $P(0, \nu) = 0$ and $\alpha(\nu, t, x, k) \leqslant \| F(w, \nu)\| \, P(t, \nu)$. This concludes our result.

4. The Paley–Wiener Theorem

We assume throughout this section that G is a linear Lie group of split rank one. Fix τ and V as in Section 2 and let $f \in C_c^\infty(G, \tau)$. Consider the map $\hat{M} \times \mathfrak{a}_{\mathbb{C}}^* \to V^M$ given by $(\omega, \nu) \to \psi_f(\omega, \nu)$ which we call the Fourier–Laplace transform of f.

Suppose now that $f(k_1 \exp Hk_2) = 0$ $(k_1, k_2 \in K)$ whenever $H \in \mathfrak{a}$ and $B(H, H) > C^2$. Normalize $\mathfrak{a}_{\mathbb{C}}^*$ by identifying $\nu \in \mathfrak{a}_{\mathbb{C}}^*$ with $\nu(H_0) \in \mathbb{C}$ where $B(H_0, H_0) = 1$ with $H_0 . \mathfrak{a}$. In this section we give a characterization of the support of f in terms of its Fourier–Laplace transform and thus give an analog of the classical Paley–Wiener theorem.

For $g \in G$ let us write $\| g \| = B(H, H)^{1/2}$ where $g = k_1 \exp Hk_2(k_1, k_2 \in K, H \in \mathfrak{a})$. Then if $a \in A$ and $n \in N \, \| an \| \geqslant \| a \|$ and $\{n \in N: \| an \| \leqslant C\}$ is compact (see [5]). This guarantees that the function $F_f: K \times A \to V$ defined by

$$F_f(k, a) = \int_N f(kan)\, dn$$

has the property that $F_f \in C^\infty(K \times A, V)$ and $F_f(k, a) = 0$ if $\| a \| > C$.

For f as above observe that the map $\nu \to \psi_f(\omega, \nu)$ satisfies the following conditions:

(1) The map $\nu \to \psi_f(\omega, \nu)$ is holomorphic and for any integer $N \geqslant 0$ there is a constant $C_N > 0$ for which

$$\| \psi_f(\omega, \nu)\| \leqslant C_N(1 + | \nu |)^{-N} e^{A|\mathrm{Im}\, \nu|}.$$

(2) Observe that $\psi_f(\omega, \nu) \in V^M(\omega)$ where $V^M(\omega)$ is the image of V^M under the map $E_\omega = d(\omega) \int_M \overline{\chi_\omega(m)}\, \tau(m)\, dm$.

Then $C(s : \nu)(\psi_f(\omega, \nu)) \in V^M(s\omega)$ but we can say more. The Eisenstein integral $E(\psi_f(w, \nu) : \nu : x)$ is in fact equal to

$$\theta_{\omega,\nu}(L_{x^{-1}} f) = \int f(xy)\, \overline{\theta_{\omega,\nu}(y)}\, dy$$

where $\theta_{\omega,\nu}$ is the character of a certain representation and for $s \in W$, $\theta_{\omega,\nu} = \theta_{s\omega, s\nu}$. Thus we have

$$E(\psi_f(\omega, \nu) : \nu : x) = E(\psi_f(s\omega, s\nu) : s\nu : x).$$

From Harish-Chandra [4] we see that for $t \in W$

$$C(t : s\nu)(\psi_f(s\omega, s\nu)) = C(ts : \nu)(\psi_f(\omega, \nu))$$

or equivalently

$$C(1 : s\nu)(\psi_f(s\omega, s\nu)) = C(s : \nu)(\psi_f(\omega, \nu)).$$

Remark. Observe that (2) implies that for $s, t \in W$

$$C^0(st : \nu)\, C(1 : \nu)(\psi_f(\omega, \nu)) = C^0(s : t\nu)\, C^0(t : \nu)\, C(1 : \nu)(\psi_f(\omega, \nu))$$

where $C^0(s : \nu) = C(s : \nu)\, C(1 : \nu)^{-1}$. Harish-Chandra has in fact shown that

$$C^0(st : \nu) = C^0(s : t\nu)\, C^0(t : \nu).$$

Besides (1) and (2) there is another discrete criterion which must be met which we now explain. Recall that

$$f(g) - \sum_{\omega \in \hat{M}} \int_{\mathfrak{a}^*} E(\psi_f(\omega, \nu) : \nu : g)\, \mu(\omega : \nu)\, d\nu$$

is a cusp form on G, and that for $a = \exp H \in \mathfrak{a}^+$, $\nu \in \mathfrak{a}^* \sim \{0\}$

$$E(\psi_f(\omega, \nu) : \nu : x) = E_1(a : \nu)\, C(1 : \nu)(\psi_f(\omega, \nu)) + E_{-1}(a : \nu)\, C(-1 : \nu)(\psi_f(\omega, \nu))$$

where E_s has the expansion defined in Section 2.

The Maass–Selberg relations of Harish-Chandra [4] say that for $\nu \in \mathfrak{a}^*$

$$C(s : \nu)\, C(s : \nu)^* = C(s : \nu)^* C(s : \nu) = \mu(\omega : \nu)^{-1} d(\omega) I$$

on $V^M(\omega)$. Select $\epsilon > 0$ so that $\mu(\omega : \nu)$ and all $\Gamma_\mu(i\nu - \rho)$ are analytic for $0 < \|\operatorname{Im} \nu\| < \epsilon$. Then by contout integration we see that for $0 < \eta < \epsilon$

$$\int_{\mathfrak{a}^*} E(\psi_f(\omega, \nu) : \nu : g)\, \mu(\omega : \nu)\, d\nu = \int_{\operatorname{Im}\nu=\eta} E(\psi_f(\omega, \nu) : \nu : g)\, \mu(\omega : \nu)\, d\nu,$$

$$f^+(a) = \sum_{\omega\in\hat{M}} \int_{\operatorname{Im}\nu=\eta} E_1(a : \nu) \circ C(1 : \nu)(\psi_f(\omega, \nu))\, \mu(\omega : \nu)\, d\nu,$$

and

$$f^-(a) = \sum_{\omega\in\hat{M}} \int_{\operatorname{Im}\nu=\eta} E_{-1}(a : \nu) \circ C(-1 : \nu)(\psi_f(\omega, \nu))\, \mu(\omega : \nu)\, d\nu.$$

Lemma 2.3, (as in Helgason [5]) guarantees that

$$f^+(a) = \sum_{\omega\in\hat{M}} \sum_{\mu\in L} d(\omega)^{-1} \int_{\operatorname{Im}\nu=\eta} \Gamma_\mu(i\nu - \rho) \circ \bar{C}(1 : \nu)^{-1}(\psi_f(\omega, \nu))\, e^{(i\nu-\rho-\mu)(H)}\, d\nu$$

and that

$$f^-(a) = \sum_{\omega\in\hat{M}} \sum_{\mu\in L} d(\omega)^{-1} \int_{\operatorname{Im}\nu=\eta} \Gamma_\mu(-i\nu - \rho) \circ \bar{C}(-1 : \nu)^{-1}(\psi_f(\omega, \nu))\, e^{-(i\nu+\rho+\mu)(H)}\, d\nu.$$

By our results of Sections 2 and 3 there is an $R > 0$ such that for $\operatorname{Im}\nu \geqslant R$, $\bar{C}(1 : \nu)$ is invertible and $\Gamma_\mu(i\nu - \rho)$ is nonsingular. Let $\nu_1, \ldots, \nu_r$ be the points $\eta < \operatorname{Im}\nu_i < R$ at which $\bar{C}(1 : \nu)^{-1}$ or $\Gamma_\mu(i\nu - \rho)$ have singularities. Then we obtain

$$f^-(a) = \operatorname{Res}_0(f)(a) + f^-_{-\eta}(a)$$

where

$$f^-_{-\eta}(a) = \sum_{\omega\in\hat{M}} \sum_{\mu\in L} d(\omega)^{-1} \int_{\operatorname{Im}\nu=-\eta} (\Gamma_\mu(-i\nu - \rho) \circ \bar{C}(-1 : \nu)^{-1}(\psi_f(\omega, \nu))\, e^{-(i\nu+\rho+\mu)(H)}\, d\nu$$

and

$$\operatorname{Res}_0(f^-)(a)$$
$$= 2\pi i \sum_{\omega\in\hat{M}} \sum_{\nu\in L} d(\omega)^{-1} \operatorname{Res}(\Gamma_\nu(-i\nu - \rho) \circ \bar{C}(-1 : \nu)^{-1}(\psi_f(\omega, \nu))\, e^{-(i\nu+\rho+\mu)(H)}, 0).$$

From (2) we see that $f^-_{-\eta}(a) = f^+(a)$ and thus we obtain

$$\int_{\mathfrak{a}^*} E(\psi_f(\omega, \nu) : \nu : a)\, \mu(\omega : \nu)\, d\nu = \operatorname{Res}_0(f)(a) + 2f^+(a).$$

Now by our results of Sections 2 and 3 there is an $R > 0$ such that for $\operatorname{Im}\nu \geqslant R\bar{C}(1 : \nu)$ is invertible and $\Gamma_\mu(i\nu - \rho)$ is nonsingular. Letting $\nu_1, \ldots, \nu_r$ be the points $\eta < \operatorname{Im} \nu_i < R$ at which $\bar{C}(1 : \nu)^{-1}$ or $\Gamma_\mu(i\nu - \rho)$ have singularities we then obtain

$$2f^+(a) = 2 \operatorname{Res}_1(f)(a) + 2fR(a)$$

where

$$f_\epsilon(a) = \sum_{\omega \in \hat{M}} \sum_{\mu \in L} d(\omega) \int_{\operatorname{Im} \nu = R} L_\mu(i\nu - \rho) \circ C(1 : \nu)^{-1}(\psi_f(\omega, \nu))\, e^{(i\nu - \rho - \mu)(H)}\, d\nu,$$

$\operatorname{Res}_1(f)(a)$

$$= -2\pi i \sum_{j=1}^{r} \sum_{\omega \in \hat{M}} \sum_{\mu \in L} d(\omega) \operatorname{Res}(\Gamma_\mu(i\nu - \rho) \circ C(1 : \nu)^{-1}(\psi_f(\omega, \nu))\, e^{(i\ -\rho - \mu)(H)}, \nu_j).$$

$\operatorname{Res}(f) = \operatorname{Res}_0(f) + 2 \operatorname{Res}_1(f)$ is $\mathscr{Z}$-finite away from the identity of A, and thus extends to a $\mathscr{Z}$-finite function off of the identity of G.

Our results of Sections 2 and 3 guarantee for $\operatorname{Im} \nu > R$ and N a positive integer the existence of a constant $C_N > 0$ for which

$$\| \Gamma_\mu(i\nu - \rho) \circ C(1 : \nu)^{-1}(\psi_f(\omega, \nu))\| \leqslant C_N(1 + |\nu|)^{-N} e^{C|\operatorname{Im} \nu|}$$

Choosing N sufficiently large that $\alpha = \int_{\mathfrak{a}^*} (1 + |\nu|)^{-N}\, d\nu < \infty$ we see that for $\eta > R$

$$\int_{\operatorname{Im} \nu = \eta} \| \Gamma_\mu(i\nu - \rho) \circ C(1 : \nu)^{-1}(\psi_f(\omega, \nu))\, e^{(i\nu - \rho - \mu)(H)} \|\, d\nu$$
$$\leqslant C_N \alpha e^{(C - B(H_0, H))\eta}$$

Letting $\eta \to \infty$ we see that $f_\epsilon(a) = 0$ for $a = \exp H$ and $B(H, H) > C^2$.

(3) From the Plancherel formula for G we have that $f(a) = 2f_\epsilon(a) + \operatorname{Res} f(a) + H(a)$ where H is a cusp form on G. Since $f(a) = 0 = 2f_\epsilon(a) = 0$ for $\| a \| > C$ we see that $\operatorname{Res} f(a) = -H(a)$ for $\| a \| > C$. Since $\operatorname{Res} f(a) + H(a)$ is $\mathscr{Z}$-finite on A' and hence analytic we see that $\operatorname{Res} f$ extends to a cusp form on G and $f(a) = 2f_\epsilon(a)$.

Now consider a function F: $\hat{M} \times \mathfrak{a}_{\mathbb{C}}^* \to V^M$ such that $F(\omega, \nu) \in V^M(\omega)$.

Definition 4.1. We say that F has condition $P_r{}^C$ if F satisfies the following conditions.

(1) For $\omega \in \hat{M}$ the map $\nu \to F(\omega, \nu)$ is holomorphic and for any integer $N \geqslant 0$ there is a constant $C_N > 0$ such that

$$\| F(\omega, \nu)\| \leqslant C_N(1 + |\nu|)^{-N} e^{C\|\operatorname{Im} \nu\|}.$$

(2) For $s \in W$ $C(s : \nu)(F(\omega, \nu)) = C(1 : s\nu)(F(s\omega, s\nu))$.

(3) The function

$$\operatorname{Res}(f)(a)$$

$$= -2\pi i \sum_{\omega\in\hat{M}} \sum_{\mu\in L} d(\omega)^{-1} \operatorname{Res}(\Gamma_\mu(i\nu-\rho)\circ \bar{C}(1:\nu)^{-1}(F(\omega,\nu))\, e^{(i\nu-\rho-\mu)(H)}, 0)$$

$$-4\pi i \sum_{\operatorname{Im}\nu\geqslant 0} \sum_{\omega\in\hat{M}} \sum_{\mu\in L} d(\omega)^{-1} \operatorname{Res}(\Gamma_\mu(i\nu-\rho)\circ \bar{C}(1:\nu)^{-1}(F(\omega,\nu))\, e^{(i\nu-\rho-\mu)(H)}, \nu)$$

where $a = \exp H$ is the restriction to A^+ of a cusp form on G.

Observe from (2) of Definition 4.1 that

$$-\sum_{\omega\in\hat{M}} d(\omega)^{-1} \operatorname{Res}(\Gamma_\mu(i\nu-\rho)\circ \bar{C}(1:\nu)^{-1}(F(\omega,\nu))\, e^{i\nu(H)}, 0)$$

$$= \sum_{\omega\in\hat{M}} d(\omega)^{-1} \operatorname{Res}(\Gamma_\mu(-i\nu-\rho)\circ \bar{C}(-1:\nu)^{-1}F(\omega,\nu)\, e^{-i\nu(H)}, 0)$$

We now prove the following analog of the classical Paley–Wiener theorem.

THEOREM 4.1. *Let G be a linear Lie group of split rank one. Suppose $f\in C_c(G,\tau)$ and $f(\exp H) = 0$ for $B(H,H) > C^2$ with $H\in\mathfrak{a}$. Then $F(\omega,\nu) = \psi_f(\omega,\nu)$ has condition $P_\tau{}^C$. Conversely, if $F(\omega,\nu)$ has condition $P_\tau{}^C$ then the function*

$$f(g) = \sum_{\omega\in\hat{M}} \int_{\mathfrak{a}^*} E(F(\omega,\nu):\nu:g)\,\mu(\omega:\nu)\,d\nu$$

is C^∞ and there is a function $J\in C_c{}^\infty(G,\tau)$ for which $J(\exp H) = 0$ if $B(H,H) > C^2$ with $H\in\mathfrak{a}$ and $f - J$ is a cusp form on G. Moreover, $\psi_f = \psi_J = F$.

Proof. If $f\in C_c{}^\infty(G,\tau)$ and $f(\exp H) = 0$ for $B(H,H) > C^2$ with $H\in A$ that $F(\omega,\nu) = \psi_f(\omega,\nu)$ has condition $P_\tau{}^c$ has already been proved.

Suppose now that F has condition $P_\tau{}^c$, then the function

$$f(g) = \sum_{\omega\in\hat{M}} \int_{\mathfrak{a}^*} E(F(\omega,\nu):\nu:g)\,\mu(\omega:\nu)\,d\nu$$

is C^∞ by Lemma 3.5 and the existence of the function J is now guaranteed by our previous remarks. Since $f - J$ is a cusp form in G $\psi_f = \psi_J$.

Now set $H(\omega,\nu) = \psi_f(\omega,\nu) - F(\omega,\nu)$. Then H has condition $P_\tau{}^c$ and

$$h(g) = \sum_{\omega\in\hat{M}} \int_{\mathfrak{a}^*} E(H(\omega,\nu):\nu:g)\,\mu(w:\nu)\,d\nu \equiv 0.$$

In order to complete the proof that $H = 0$ we use the following formula which is due to Harish-Chandra

$$E(\psi_h(\omega', \nu) : \nu : 1) = \sum_{\omega \in \hat{M}} \sum_{s \in W} E_{\omega'}(\mathrm{s}C(s : s^{-1}\nu)(H(\omega, s^{-1}\nu)))$$

where ${}^\circ C(s : s^{-1}\nu) = C(1 : \nu)^{-1} C(s : s^{-1}\nu)$.

Now since H has condition $P_\tau{}^c$ we have that $C(s : s^{-1}\nu)\, H(\omega, s^{-1}\nu) = C(1 : \nu)\, H(s\omega, \nu)$ and hence ${}^\circ C(s : s^{-1}\nu)(H(\omega, s^{-1}\nu)) = H(s\omega, \nu)$. Thus summing we have that if $H \neq 0$, $\psi_k \neq 0$ and hence $h \neq 0$.

COROLLARY. *Suppose $f \in C_c{}^\infty(G, \tau)$ and for any integer $N \geqslant 0$ there is a constant $C_N > 0$ such that*

$$\| \psi_f(\omega, \nu)\| \leqslant C_N(1 + |\nu|)^{-N} e^{C|\mathrm{Im}\,\nu|}.$$

Then $f(\exp H) = 0$ for $H \in \mathfrak{a}$ and $B(H, H) > C^2$.

Remarks. (1) The formulation we have given our analog of the Paley–Wiener theorem is in the same nature as that first formulated by Helgason in [5].

(2) For a more detailed study of condition (3) of $P_\tau{}^C$ for $SU(1, 1)$ we refer to [16]. For complex groups we refer to Zelobenko [15].

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